Lecture Notes on Partial Differential Equations (PDE)/ MaSc 221+MaSc 225

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Preface

This lecture notes have been designed for the use of undergraduate students inrolled in MaSc 221 and MaSc 225 in mathematical sciences department in Princess Nourah bint Abdulrahman University .The material that this lecture notes covers can be viewed as a first course on the solution of partial differential equations of first order. If you have any comments , I would appreciate your contacting me at e-mail address (aialthumairi@pnu.edu.sa).

Contents

1 INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

1.1 Introductory Concepts

Def 1. *A partial differential equation (PDE) is a differential equation in which the unknown function depends on two or more independent variables. For example,*

$$u_x - 3u_y = 0$$

is a PDE in which u is the (unknown) dependent variable, while x and y are the independent variables.

We can express a PDE in one dependent variable and two independent variables by the general form

$$f(x, y, z, z_x, z_y, z_{xy}, z_{xx}, z_{yy}, \dots) = 0 \tag{1}$$

z is (unknown) dependent variable. x, y are independents variables.

Notations

Partial derivatives are often denoted by **subscript notation** indicating the independent variables.For example,

$$z_x = \frac{\partial z}{\partial x}, z_y = \frac{\partial z}{\partial y}, z_{xy} = \frac{\partial^2 z}{\partial x \partial y}, z_{xx} = \frac{\partial^2 z}{\partial x^2}, z_{yy} = \frac{\partial^2 z}{\partial y^2}$$

We adopt the following notations throughout the study of PDEs,

$$p = z_x, q = z_y, r = z_{xx}, s = z_{xy}, t = z_{yy}$$

Example 1.1.1. *Write the following equations by using the previous symbols.*

$$z_{xy} + 2z_x + 3z_y + 5z = 2x + \cos(x - y)$$

$$z_x + 3z_y = 5z + \tan(3x - 2y)$$

$$z_{xx} + z_{yy} = x^2 y^2$$

1.2 Classification of PDE

To talk about PDE(s), we shall classify them by **order, number of variables, linearity, and types of coefficients**

1.2.1 Classification by order

Def 2. *The order of a PDE is the order of the highest derivative appearing in the equation.*

The degree of a PDE is the power of highest derivative appearing in the equation.

Example 1.2.1.

$$z_t = z_{xx}$$

is a second-order PDE.

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

is a first-order PDE.

$$z_t = z_{xxx} + \sin x$$

is a third-order PDE.

1.2.2 Classification by number of variables

The number of variables is the number of an independent variables appearing in the equation. For example, $z_t = z_{xx}$ has two independent variables (t, x), and $u_t = u_{rr} + \frac{1}{r}u_{\theta\theta}$ has three independent variables (t, r, θ)

1.2.3 Classification by linearity

Def 3. A linear n-th order PDE *is a PDE which can be put in the form. The left side of the equation is a linear combination of the unknown function z and its partial derivative (up to order n) with coefficients which are given functions of the independent variables or constants. The right side must be some given function f of the independent variables. If function f is indentically zero, then the linear PDE is called a* **homogeneous** *PDE.*

Example 1.2.2. *Classify the following PDE(s) as linear or nonlinear, specify whether it is homogeneous or inhomogeneous.*

(a) $x^2 u_{xxy} + y^2 u_{yy} - \ln(1 + y^2)u = 0$ **homogeneous linear PDE.**

Nonlinear functions of the dependent variable or its derivative, such as $\sin u, \cos u, e^{u'}$, or $\ln u$, cannot appear in a linear equation.

(b) $u_x + u^3 = 1$ **nonhomogeneous nonlinear PDE.**

(c) $uu_{xx} + u_{yy} - u = 0$ **homogeneous nonlinear PDE.**

(d) $u_{xx} + u_t = 3u$ **homogeneous linear PDE.**

(e) $u_{xxyy} + e^x u_x = y$ **nonhomogeneous linear PDE.**

Remark. *The general second-order linear PDE for an unknown function $u = u(x, y)$ is*

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{2}$$

where A, B, C, D, E, F and G are given functions (possibly constants) of x and y, with $A, B,$ and C not all zero. If $G \equiv 0$, then (2) is the general second-order homogeneous linear PDE.

1.2.4 Classification by types of coefficients

- If A, B, C, D, E, F and G in (2) are constants then it called linear PDE with **constants coefficients**.
- If one or more of the coefficients A, B, C, D, E, F and G in (2) are functions of x or y or x, y then it called linear PDE with **variable coefficients**.

1.3 Classification of second-order linear PDE

If we combine the lower order terms and rewrite (2) in the following form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + I(x, y, u, u_x, u_y) = 0 \tag{3}$$

As we will see, the type of the above equation depends on the sign of the quantity

$$\Delta(x, y) = B^2(x, y) - 4A(x, y)C(x, y), \tag{4}$$

Which is called the discriminant for (3). The classification of second-order linear PDE is given as following

Def 4. *At the point (x_0, y_0) the second-order linear PDE (3) is called*

(i) hyperbolic, if $\Delta(x_0, y_0) > 0$

(ii) parabolic, if $\Delta(x_0, y_0) = 0$

(iii) elliptic, if $\Delta(x_0, y_0) < 0$

Example 1.3.1. *Classify the following PDEs as parabolic, elliptic, and hyperbolic.*

(a) $u_t = u_{xx}$
$u_{xx} - u_t = 0$
$A = 1, B = 0, C = 0$
$\Delta = B^2 - 4AC = 0 - 4(0) = 0$, *the PDE is parabolic.*

(b) $u_{tt} = u_{xx}$
$u_{xx} - u_{tt} = 0$
$A = 1, B = 0, C = -1$
$\Delta = B^2 - 4AC = 0 - (1)(-1) = 4 > 0$, *the PDE is hyperbolic*

(c) $u_{xy} = 0$
$A = 0, B = 1, C = 0$
$\Delta = B^2 - 4AC = 1 - 4(0) = 1 > 0$, *the PDE is hyperbolic*

(d) $\alpha u_{xx} + u_{yy} = 0$, *where* α *is a constant,.*
$A = \alpha, B = 0, C = 1$
$\Delta = B^2 - 4AC = 0 - 4\alpha = -4\alpha$

We have three properties,

(i) if $\alpha = 0$ *the PDE is parabolic.*

(ii) if $\alpha > 0$ *the PDE elliptic.*

(iii) if $\alpha < 0$ *the PDE is hyperbolic.*

1.4 Solutions and Solution Techniques

1.4.1 Applications of PDEs

There are many applications of PDEs, in our study we will adopt the most important PDEs that arise in various branches of science and engineering.

Heat Equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \boxed{k \text{ is a positive constant.}} \tag{5}$$

Wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k^2}\frac{\partial^2 u}{\partial t^2}, \boxed{k \text{ is a positive constant.}} \tag{6}$$

Laplace's equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \tag{7}$$

Def 5. *A function $u(x, y)$ is called* **harmonic** *if it satisfies Laplace's equation; that is $u_{xx} + u_{yy} = 0$.*

Example 1.4.1. *Verify that $u = e^{3x} \cos 3y$ is a harmonic function.*

Solution. *Taking the derivatives of u leads us to:*
$u_x = 3e^{3x} \cos 3y, \quad u_{xx} = 9e^{3x} \cos 3y$
$u_y = -3e^{3x} \sin 3y, \quad u_{yy} = -9e^{3x} \cos 3y$
$\therefore u_{xx} + u_{yy} = 9e^{3x} \cos 3y - 9e^{3x} \cos 3y = 0$, *u indeed is a harmonic function.*

1.4.2 Solution of linear PDEs

Def 6. *A solution of linear PDE (2) is a function $u = g(x, y)$ that satisfies the differential equation.*

Example 1.4.2. *Verify that $u(x,t) = \sin x \cos kt$ satisfies the wave equation (6).*

Solution. *Taking derivatives of u leads us to :*
$u_x = \cos x \cos kt, \quad u_{xx} = -\sin x \cos kt,$
$u_t = -k \sin x \sin kt, \quad u_{tt} = -k^2 \sin x \cos kt.$
$\because u_{xx} = \frac{1}{k^2} u_{tt}$
$\therefore -\sin x \cos kt = \frac{1}{k^2}(-k^2 \sin x \cos kt) = -\sin x \cos kt$, *u indeed is a solution.*

Example 1.4.3. *Verify that $u(x, y) = x^2 - y^2$ satisfies Laplace's equation(47).*

Solution. *Taking derivatives of u leads us to:*
$u_x = 2x, \quad u_{xx} = 2,$
$u_y = -2y, \quad u_{yy} = -2.$
$\because u_{xx} + u_{yy} = 0$
$\therefore 2 - 2 = 0$; *u indeed is a solution.*

Example 1.4.4. *Verify that any function of the form $F(x + kt)$ satisfies the wave equation (6).*

Solution. *Let $u = x + kt$; then by using the chain rule for partial derivatives, we have:*
$F_x = \frac{dF}{du} u_x = \frac{dF}{du}(1) = \frac{dF}{du}, \quad F_{xx} = \frac{d^2F}{du^2} u_x = \frac{d^2F}{du^2}(1) = \frac{d^2F}{du^2}$
$F_t = \frac{dF}{du} u_t = \frac{dF}{du}(k), \quad F_{tt} = k\frac{d^2F}{du^2} u_t = k^2 \frac{d^2F}{du^2}.$

$\because F_{xx} = \frac{1}{k^2} F_{tt}$
$\therefore \frac{d^2F}{du^2} = \frac{1}{k^2}(k^2 \frac{d^2F}{du^2}) = \frac{d^2F}{du^2}$, *so we have verified that any sufficiently differentiable function of the form $F(x+kt)$ satisfies the wave equation. We note that this means that functions such as $\sqrt{x+kt}$, $\tan^{-1}(x+kt)$ and $\ln(x+kt)$ all satisfy the wave equation.*

Example 1.4.5. *Verify that $u(x,t) = e^{-kt}\sin x$ satisfies the heat equation (5).*

Solution. *Taking derivatives of u leads us to:*
$u_x = e^{-kt}\cos x, \quad u_{xx} = -e^{-kt}\sin x$
$u_t = -ke^{-kt}\sin x.$
$\because u_{xx} = \frac{1}{k}u_t$
$\therefore -e^{-kt}\sin x = \frac{1}{k}(-ke^{-kt}\sin x) = -e^{-kt}\sin x$, *u indeed is a solution.*

1.4.3 Solving linear PDEs by basic integration

Example 1.4.6. *Let $z = z(x,y)$. By integration, find the general solution to $z_x = 2xy$.*

Solution. *Integrating with respect to x, we have:*
$\int z_x \, dx = \int 2xy \, dx$
$z(x,y) = x^2y + g(y)$, *where $g(y)$ is any differentiable function of y.*

Example 1.4.7. *Let $u = u(x,y)$. By integration, find the general solution to $u_x = 0$.*

Solution. *Integrating with respect to x, we have:*
$\int u_x \, dx = \int 0 \, dx$
$u(x,y) = g(y)$, *where $g(y)$ is any differentiable function of y.*

Example 1.4.8. *Let $u = u(x,y,z)$. By integration, find the general solution to $u_x = 0$.*

Solution. *Integrating with respect to x, we have:*
$\int u_x \, dx = \int o \, dx$
$u(x,y,z) = f(y,z)$, *where $f(y,z)$ is any differentiable function of y, z.*

Example 1.4.9. *Let $u = u(x,y)$. By integration, find the general solution to $u_x = 2x, u(0,y) = \ln y$.*

Solution. *Integrating with respect to x, we have:*
$\int u_x \, dx = \int 2x \, dx$
$u(x,y) = x^2 + f(y)$, *where $f(y)$ is any differentiable function of y.*
Letting $x = 0$ implies $u(0,y) = 0^2 + f(y) = \ln y$. Therefore $f(y) = \ln y$, so our solution is $u(x,y) = x^2 + \ln y$.

Example 1.4.10. *Let* $u = u(x, y)$*. By integration, find the general solution to* $u_y = 2x$*.*

Solution. *Integrating with respect to* y*, we have:*
$\int u_y \, dy = \int 2x \, dy$
$u(x, y) = 2xy + g(x)$*, where* $g(x)$ *is any differentiable function of* x*.*

Example 1.4.11. *Let* $u = u(x, y)$*. By integration, find the general solution to* $u_{xy} = 2x$*.*

Solution. *Integrating first with respect to* x*, we have:*
$\int u_{xy} \, dx = \int 2x \, dx$
$u_y = x^2 + g(y)$*, where* $g(y)$ *is any differentiable function of* y*. We now integrate* u_y *with respect to* y*.*
$\int u_y \, dy = \int (x^2 + g(y)) \, dy$,
$u(x, y) = x^2 y + f(y) + h(x)$*, where* $f(y)$ *is an antiderivative of* $g(y)$*, and* $h(x)$ *is any differentiable function of* x*.*

If we solve $u_{yx} = 2x$**, our result would be the same.**

Supplementry Problems

1. Verify that any function of the form $F(x - kt)$ satisfies the wave equation (6).
2. If $u = f(x - y)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$.
3. Which of the following functions are harmonic: $(a) 3x + 4y + 1$; $(b) e^{3x} \cos 4y$; $(c) \ln(x^2 + y^2)$; $(d) \sin(e^x) \cos(e^y)$.
4. Find the general solution to $u_x = \cos y$ if $u(x, y)$ is a function of x and y.
5. Find the general solution to $u_y = \cos y$ if $u(x, y)$ is a function of x and y.
6. Find the general solution to $u_y = 3$ if $u(x, y)$ is a function of x and y, and $u(x, 0) = 4x + 1$.
7. Find the general solution to $u_x = 2xy + 1$ if $u(x, y)$ is a function of x and y, and $u(0, y) = \cosh y$.
8. Find the general solution to $u_{xy} = 8xy^3$ if $u(x, y)$ is a function of x and y.
9. Find the general solution to $u_{xx} = 3$ if $u(x, y)$ is a function of x and y.

2 Formation of PARTIAL DIFFERENTIAL EQUATIONS by Elimination

We shall now examine the interesting question of how partial differential equation arise. We show that such equations can be formed by the elimination of arbitrary constants or arbitrary functions.

2.1 Derivation of PDE by the elimination of arbitrary constants.

Consider an equation

$$F(x, y, z, a, b) = 0 \tag{8}$$

where a and b denote arbitrary constants. Let z be regarded as function of two independent variables x and y. Differentiating (8) with respect to x and y partially in turn, we get

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0, \quad \frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0 \tag{9}$$

Eliminating two constants a and b from equations (8) and (9), we shall obtain an equation of the form

$$f(x, y, z, p, q) = 0 \tag{10}$$

which is partial differential equation of first order.

In similar manner it can be shown that if there are more arbitrary constants than number of independent variable, the above procedure of elimination will give rise to PDEs of higher than the first.

Example 2.1.1. *Construct the PDE by eliminating a and b from*

$$z = ax + (1 - a)y + b \tag{11}$$

Solution. *Differentiating* (11) *with respect to x, y , we get*

$$p = \frac{\partial z}{\partial x} = a \tag{12}$$

$$q = \frac{\partial z}{\partial y} = 1 - a \tag{13}$$

Now finding the sum of (12) *and* (13), *we get*

$$p + q = 1 \tag{14}$$

Example 2.1.2. *Find a PDE by eliminating the arbitrary constants from*

$$z = ax^2 + by^2, \quad ab > 0 \tag{15}$$

Solution. *Differentiating* (15) *with respect to* x, y *, we get*

$$p = \frac{\partial z}{\partial x} = 2ax \longrightarrow a = \frac{p}{2x} \tag{16}$$

$$q = \frac{\partial z}{\partial y} = 2by \longrightarrow b = \frac{q}{2y} \tag{17}$$

Substituting (16) *and* (17) *into* (15), *we get*

$$px + qy = 2z \tag{18}$$

Exercise. *Find a PDE by eliminating the arbitrary constants from* $z = ax^2 + by^2 + ab$

Example 2.1.3. *Construct the PDE by eliminating* a, b *and* c *from*

$$z = a(x + y) + b(x - y) + abt + c \tag{19}$$

Solution. *Differentiating* (19) *with respect to* x, y *and* t, *we get*

$$\frac{\partial z}{\partial x} = a + b, \quad \frac{\partial z}{\partial y} = a - b, \quad \frac{\partial z}{\partial t} = ab \tag{20}$$

Since $(a + b)^2 - (a - b)^2 = 4ab$ *, we get by using* (20)

$$(\frac{\partial z}{\partial x})^2 - (\frac{\partial z}{\partial y})^2 = 4\frac{\partial z}{\partial t}$$

Example 2.1.4. *Eliminate the arbitrary constant* a *from the following equation*

$$z = a(x + y) \tag{21}$$

Solution. *Differentiating* (21) *with respect to* x *or* y *, we get*

$$p = \frac{\partial z}{\partial x} = a \quad or \quad q = \frac{\partial z}{\partial y} = a \tag{22}$$

Substitute a *in* (21), *we get*

$$z = p(x + y) \quad or \quad z = q(x + y)$$

Example 2.1.5. *Construct the PDE by eliminating the arbitrary constants from*

$$z = (x^2 + a)(y^2 + b) \tag{23}$$

Solution. *Differentiating* (23) *with respect to* x, y *, we get*

$$p = \frac{\partial z}{\partial x} = 2x(y^2 + b) \longrightarrow (y^2 + b) = \frac{p}{2x} \tag{24}$$

$$q = \frac{\partial z}{\partial y} = 2y(x^2 + a) \longrightarrow (x^2 + a) = \frac{q}{2y} \tag{25}$$

Substitute (24) *and* (25) *in* (23), *we get*

$$pq = 4xyz.$$

Example 2.1.6. *Construct the PDE by eliminating the arbitrary constants from*

$$z = ax + by + cxy \tag{26}$$

Solution. *Differentiating* (26) *with respect to* x, y *, we get*

$$p = \frac{\partial z}{\partial x} = a + cy \longrightarrow a = p - cy \tag{27}$$

$$q = \frac{\partial z}{\partial y} = b + cx \longrightarrow b = q - cx \tag{28}$$

Now differentiating (27) *with respect to* y *or* (28) *with respect to* x *, we get*

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = c \tag{29}$$

Substitute (27),(28) *and* (29) *into* (26), *we get*

$$z = px + qy - sxy$$

2.2 Derivation of PDE by the elimination of arbitrary functions.

If $\phi(u, v) = 0$ is an arbitrary function of u and v, where u and v are functions of x, y and z. We treat z as dependent variable and x and y as independent variables so that

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial x}{\partial y} = 0, \quad \frac{\partial y}{\partial x} = 0$$

Example 2.2.1. *Form a PDE by eliminating the arbitrary function f from*

$$z = f(x - y) \tag{30}$$

Solution. *Differentiating partially* (30) *with respect to x and y , we get*

$$p = \frac{\partial z}{\partial x} = f'(x - y)(1) = f'(x - y) \tag{31}$$

$$q = \frac{\partial z}{\partial y} = f'(x - y)(-1) = -f'(x - y) \tag{32}$$

Now finding the sum of (31) *and* (32), *we get* $p + q = 0$

Example 2.2.2. *Form a PDE by eliminating the arbitrary function ϕ from*

$$\phi(x + y + z, x^2 + y^2 - z^2) = 0. \tag{33}$$

Solution. *Let*

$$u = x + y + z \quad v = x^2 + y^2 - z^2 \tag{34}$$

Then (33) *becomes*

$$\phi(u, v) = 0 \tag{35}$$

Differentiating (35) *w.r.t x partially, we get*

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} = 0 \tag{36}$$

From (34), *we get*

$$\frac{\partial u}{\partial x} = 1 + p, \quad \frac{\partial v}{\partial x} = 2x - 2zp \tag{37}$$

Substituting (37) *into* (36), *we get*

$$\frac{\partial \phi}{\partial u}(1 + p) + \frac{\partial \phi}{\partial v}(2x - 2zp) = 0 \tag{38}$$

Again, diff. (35) *w.r.t. y partially, we get*

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} = 0 \tag{39}$$

From (34), *we get*

$$\frac{\partial u}{\partial y} = 1 + q, \quad \frac{\partial v}{\partial y} = 2y - 2zq \tag{40}$$

Substituting (40) *into* (39), *we get*

$$\frac{\partial\phi}{\partial u}(1+q) + \frac{\partial\phi}{\partial v}(2y - 2zq) = 0 \tag{41}$$

Now we will eliminate $\frac{\partial\phi}{\partial u}$ *and* $\frac{\partial\phi}{\partial v}$ *from* (38) *and* (41), *as follows*

$$\begin{vmatrix} 1+p & 2x-2zp \\ 1+q & 2y-2zq \end{vmatrix} = 0$$

$\therefore$ *From the last equation, we get*

$$(y+z)p - (x+z)q = x - y$$

Example 2.2.3. *Eliminate an arbitrary function from the following equation*

$$x + y + z = f(x^2 + y^2 + z^2) \tag{42}$$

Solution. *Differentiating partially w.r.t* x *and* y, (42) *gives*

$$1 + p = f'(x^2 + y^2 + z^2)(2x + 2zp) \tag{43}$$

$$1 + q = f'(x^2 + y^2 + z^2)(2y + 2zq) \tag{44}$$

Determining $f'(x^2+y^2+z^2)$ *from* (43) *and* (44) *and equating the values, we eliminate* f *and obtain*

$$(1+p)/(2x+2zp) = (1+q)/(2y+2zq)$$

or

$$(1+p)(2y+2zq) = (1+q)(2x+2zp)$$

$$\therefore (y-z)p + (z-x)q = x - y$$

Note. *If the given equation between* x, y, z *contains two arbitrary functions, then in general, their elimination gives rise to equations of higher orders.*

Example 2.2.4. *Eliminate the arbitrary functions* f *and* g *from*

$$y = f(x - at) + g(x + at) \tag{45}$$

Solution. *Differentiating partially w.r.t x and t , (45) gives*

$$\begin{aligned} \frac{\partial y}{\partial x} &= f'(x-at) + g'(x+at) \\ \frac{\partial^2 y}{\partial x^2} &= f''(x-at) + g''(x+at) \\ \frac{\partial y}{\partial t} &= f'(x-at)(-a) + g'(x+at)(a) \\ \frac{\partial^2 y}{\partial t^2} &= f''(x-at)(a)^2 + g''(x+at)(a)^2 \end{aligned} \tag{46}$$

From the last equation, we get

$$\frac{\partial^2 y}{\partial t^2} = a^2(f''(x-at) + g''(x+at))$$

$$\therefore \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Supplementry Problems

1. Eliminate the arbitrary constants indicated in brackets from the following equations and form the PDE.

   (a) $z = (x-a)^2 + (y-b)^2 \quad (a \text{ and } b)$

   (b) $z = axy + b \quad (a \text{ and } b)$

   (c) $az + b = a^2x + y \quad (a \text{ and } b)$

   (d) $z = ax + by + a^2 + b^2 \quad (a \text{ and } b)$

   (e) $ax + by + cz = 1 \quad (a\ ,\ b\ ,\ c)$

   (f) $z = ae^{bt}\sin bx \quad (a \text{ and } b)$

   (g) $z = a^2x + (y-b)^2 \quad (a \text{ and } b)$

2. Eliminate the arbitrary functions and hence obtain the PDEs.

   (a) $f(\frac{z}{x^2}, x-y) = 0$

   (b) $z = x^2 f(x-y)$

   (c) $z = f(x+ay)$

   (d) $z = f(x+iy) + F(x-iy)$

   (e) $z = f(\frac{xy}{z})$

   (f) $z = f(\frac{y}{x})$

   (g) $z = xy + f(x^2+y^2)$

   (h) $z = e^{ax+by} f(ax-by)$

   (i) $z = f(x^2+y^2+z^2, z^2-2xy) = 0$

3 Lagrange's method of solving the linear PDE of first order, namely $Pp + Qq = R$.

Theorem. *The general solution of the linear PDE*

$$Pp + Qq = R \tag{47}$$

is given by

$$\phi(u, v) = 0 \tag{48}$$

where ϕ *is an arbitrary function and*

$$u(x, y, z) = c_1, \quad v(x, y, z) = c_2 \tag{49}$$

form a solution of the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{50}$$

Note. *Equations* (50) *are called Lagrange's auxiliary (or subsidiary) equations for* (47).

3.1 Working rule for solving $Pp+Qq = R$ by Lagrange's method.

Steps 1. *Put the giving linear PDE of first order in the standard form*

$$Pp + Qq = R \tag{51}$$

Steps 2. *Write down Lagrange's auxiliary equations for* (51), *namely,*

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{52}$$

Steps 3. *Solve* (52) *by using the well known methods of previous chapters. Let* $u(x, y, z) = c_1$ *and* $v(x, y, z) = c_2$ *be two independent solutions of* (52).

Steps 4. *The general solution of* (51) *is then written in one of the following three equivalent forms:*

$$\phi(u, v) = 0, \quad u = \phi(v), \quad v = \phi(u)$$

3.2 Examples based on Working rule for solving $Pp + Qq = R$ by Lagrange's method.

There were four rules for getting two independent solutions of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

3.2.1 Examples based on Rule I

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Example 3.2.1. *Solve the given PDE*

$$2p + 3q = 1 \tag{53}$$

Solution. *Lagrange's auxiliary equations for* (53) *are*

$$\frac{dx}{2} = \frac{dy}{3} = \frac{dz}{1} \tag{54}$$

Taking the first two fractions of (54) *and re-writing, we get*

$$\begin{aligned} 3dx - 2dy = 0 \\ \therefore 3x - 2y = c_1 \end{aligned} \tag{55}$$

Now taking the last two fractions of (54)*and re-writing, we get*

$$\begin{aligned} dy - 3dz = 0 \\ \therefore y - 3z = c_2 \end{aligned} \tag{56}$$

Hence the required general solution is

$$\phi(3x - 2y, y - 3z) = 0$$

Example 3.2.2. *Solve the given PDE*

$$xp + yq = z \tag{57}$$

Solution. *Lagrange's auxiliary equations for* (57) *are*

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} \tag{58}$$

Taking the first two fractions of (58) *and re-writing, we get*

$$\begin{aligned} \frac{dx}{x} - \frac{dy}{y} &= 0 \\ \ln x - \ln y &= \ln c_1 \\ \therefore \frac{x}{y} &= c_1 \end{aligned} \tag{59}$$

Now taking the last two fractions of (58)*and re-writing, we get*

$$\begin{aligned} \frac{dy}{y} - \frac{dz}{z} &= 0 \\ \ln y - \ln z &= \ln c_2 \\ \therefore \frac{y}{z} &= c_2 \end{aligned} \tag{60}$$

Hence the required general solution is

$$\phi(\frac{x}{y}, \frac{y}{z}) = 0$$

Example 3.2.3. *Solve the given PDE*

$$zp = -x \tag{61}$$

Solution. *Lagrange's auxiliary equations for* (61) *are*

$$\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x} \tag{62}$$

Taking the first two fractions of (62) *and re-writing, we get*

$$\begin{aligned} dy &= 0 \\ \therefore y &= c_1 \end{aligned} \tag{63}$$

Now taking the first and last fractions of (62)*and re-writing, we get*

$$\begin{aligned} xdx + zdz &= 0 \\ \therefore x^2 + z^2 &= c_2 \end{aligned} \tag{64}$$

Hence the required general solution is

$$\phi(y, x^2 + z^2) = 0$$

Exercise. *Solve the PDE* $p + q = 1$.

3.2.2 Examples based on Rule II.

Example 3.2.4. *Solve the given PDE*

$$p - 2q = 3x^2 \sin(y + 2x) \tag{65}$$

Solution. *Lagrange's auxiliary equations for* (65) *are*

$$\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2 \sin(y + 2x)} \tag{66}$$

Taking the first two fractions of (66) *and re-writing, we get*

$$\begin{aligned} 2dx + dy &= 0 \\ \therefore 2x + y &= c_1 \end{aligned} \tag{67}$$

Now taking the first and last fractions of (66)*and using* (67), *we get*

$$\begin{aligned} \frac{dx}{1} &= \frac{dz}{3x^2 \sin(c_1)} \\ 3x^2 \sin(c_1)dx - dz &= 0 \\ x^3 \sin(c_1) - z &= c_2 \\ x^3 \sin(2x + y) - z &= c_2 \end{aligned} \tag{68}$$

Hence the required general solution is

$$\phi(2x + y, x^3 \sin(2x + y) - z) = 0$$

Example 3.2.5. *Solve the given PDE*

$$xyp + y^2 q = zxy - 2x^2 \tag{69}$$

Solution. *Lagrange's auxiliary equations for* (69) *are*

$$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{zxy - 2x^2} \tag{70}$$

Taking the first two fractions of (70) *and re-writing, we get*

$$\begin{aligned} \frac{dx}{x} - \frac{dy}{y} &= 0 \\ \ln x - \ln y &= \ln c_1 \\ \frac{x}{y} &= c_1 \\ x &= yc_1 \end{aligned} \tag{71}$$

Now taking the first and last fractions of (70)*and using* (71), *we get*

$$\begin{aligned}\frac{dx}{y} &= \frac{dz}{zy - 2x}\\ \frac{dx}{1} &= \frac{dz}{z - 2c_1}\\ \frac{dx}{1} - \frac{dz}{z - 2c_1} &= 0\\ x - \ln(z - 2c_1) &= c_2\\ x - \ln(z - \frac{2x}{y}) &= c_2\end{aligned} \tag{72}$$

Hence the required general solution is

$$\phi(\frac{x}{y}, x - \ln(z - \frac{2x}{y})) = 0$$

Exercise. *Solve the PDE* $z(p - q) = z^2 + (x + y)^2$.

3.2.3 Examples based on Rule III

Example 3.2.6. *Solve the given PDE*

$$(y - z)p + (x - y)q = z - x \tag{73}$$

Solution. *Lagrange's auxiliary equations for* (73) *are*

$$\frac{dx}{y - z} = \frac{dy}{x - y} = \frac{dz}{z - x} \tag{74}$$

Since $(y - z) + (x - y) + (z - x) = 0$,

$$\therefore dx + dy + dz = 0$$

Integrating the last equation, we get

$$x + y + z = c_1$$

Choosing (x, z, y) *as multipliers, we re-write* (73) *as*

$$\begin{aligned}\frac{xdx + zdy + ydz}{x(y - z) + z(x - y) + y(z - x)} &= \frac{xdx + zdy + ydz}{0}\\ \therefore xdx + zdy + ydz &= 0\\ xdx + d(yz) &= 0\end{aligned} \tag{75}$$

Integrating the last equation , we get

$$x^2 + 2yz = c_2$$

Hence the required general solution is

$$\phi(x + y + z, x^2 + 2yz) = 0$$

Example 3.2.7. *Solve the given PDE*

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} \tag{76}$$

Solution. *Choosing* (l, m, n) *as multipliers, we re-write* (76) *as*

$$\frac{ldx + mdy + ndz}{l(mz - ny) + m(nx - lz) + n(ly - mx)} = \frac{ldx + mdy + ndz}{0} \tag{77}$$
$$\therefore ldx + mdy + ndz = 0$$

Integrating the last equation , we get

$$lx + my + nz = c_1$$

Again choosing (x, y, z) *as multipliers, we re-write* (76) *as*

$$\frac{xdx + ydy + zdz}{x(mz - ny) + y(nx - lz) + z(ly - mx)} = \frac{xdx + ydy + zdz}{0} \tag{78}$$
$$\therefore xdx + ydy + zdz = 0$$

Integrating the last equation , we get

$$x^2 + y^2 + z^2 = c_2$$

Hence the required general solution is

$$\phi(lx + my + nz, x^2 + y^2 + z^2) = 0$$

Exercise. *Solve the PDE* $yp - xq = 2x - 3y$.

3.2.4 Examples based on Rule IV

Example 3.2.8. *Solve the given PDE*

$$(1+y)p+(1+x)q=z \tag{79}$$

Solution. *Lagrange's auxiliary equations for* (79) *are*

$$\frac{dx}{1+y}=\frac{dy}{1+x}=\frac{dz}{z} \tag{80}$$

Choosing $(1,1,0)$ *as multipliers, we re-write* (80) *as*

$$\frac{dx+dy+0}{(1+y)+(1+x)+0}=\frac{dx+dy}{x+y+2} \tag{81}$$

Taking the last fraction of (80) *and fraction* (81), *we get*

$$\frac{dz}{z}=\frac{dx+dy}{x+y+2} \tag{82}$$

Integrating the last equation , we get

$$\frac{z}{x+y+2}=c_1$$

Again choosing $(1,-1,0)$ *as multipliers, we re-write* (80) *as*

$$\frac{dx-dy+0}{(1+y)-(1+x)+0}=\frac{dx-dy}{y-x} \tag{83}$$

Taking the last fraction of (80) *and fraction* (83), *we get*

$$\frac{dz}{z}=\frac{dx-dy}{y-x} \tag{84}$$

Integrating the last equation , we get

$$z(y-x)=c_2$$

Hence the required general solution is

$$\phi\left(\frac{z}{x+y+2},z(y-x)\right)=0$$

Example 3.2.9. *Solve the given PDE*

$$y^2(x-y)p + x^2(y-x)q = z(x^2+y^2) \tag{85}$$

Solution. *Lagrange's auxiliary equations for* (85) *are*

$$\frac{dx}{y^2(x-y)} = \frac{dy}{x^2(y-x)} = \frac{dz}{z(x^2+y^2)} \tag{86}$$

Taking the first two fractions of (86) *and re-writing, we get*

$$\begin{aligned} \frac{dx}{y^2(x-y)} &= -\frac{dy}{x^2(x-y)} \\ x^2dx + y^2dy &= o \\ \therefore x^3 + y^3 &= c_1 \end{aligned} \tag{87}$$

Choosing $(1, -1, 0)$ *as multipliers, we re-write* (86) *as*

$$\frac{dx - dy + 0}{y^2(x-y) + x^2(x-y) + 0} = \frac{dx - dy}{(x-y)(x^2+y^2)} \tag{88}$$

Taking the last fraction of (86) *and fraction* (88), *we get*

$$\begin{aligned} \frac{dz}{z(x^2+y^2)} &= \frac{dx - dy}{(x-y)(x^2+y^2)} \\ \frac{dz}{z} &= \frac{dx - dy}{(x-y)} \end{aligned} \tag{89}$$

Integrating the last equation, we get

$$\frac{z}{x-y} = c_2$$

Hence the required general solution is

$$\phi\left(x^3 + y^3, \frac{z}{x-y}\right) = 0$$

Supplementry Problems

Solve the following PDE(s) by using Lagrange's auxiliary equation.

1. $p + q = z$
2. $3p + 4q = 2$
3. $yq - xp = z$
4. $xzp + yzq = xy$
5. $x^2p + y^2q = z^2$
6. $p + q = y$
7. $(\tan x)p + (\tan y)q = \tan z$
8. $p + 3q = 5z + \tan(y - 3x)$
9. $yp + xq = xyz^2(x^2 - y^2)$

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